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On the solution near the critical frequency for an oscillating and translating body in or near a free surface

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We consider a floating or submerged body in deep water translating parallel to the undisturbed free surface with a steady velocity U while undergoing small oscillations at frequency ω. It is known that for a single source, the solution becomes singular at the resonant frequency given by $\tau \equiv U\omega/g=\frac{1}{4}$, where g is the gravitational acceleration. In this paper, we show that for a general body, a finite solution exists as $\tau \rightarrow \frac{1}{4}$ *if and only if* a certain geometric condition (which depends only on the frequency ω but *not* on U) is satisfied. For a submerged body, a necessary and sufficient condition is that the body must have non-zero volume. For a surface-piercing body, a sufficient condition is derived which has a geometric interpretation similar to that of John (1950). As an illustration, we provide an analytic (closed-form) solution for the case of a submerged circular cylinder oscillating near $\tau = \frac{1}{4}$. Finally, we identify the underlying difficulties of existing approximate theories and numerical computations near $\tau = \frac{1}{4}$, and offer a simple remedy for the latter.

1. Introduction

The oscillatory motion of a translating body in the presence of a free surface is a problem of fundamental theoretical interest. For small motion amplitudes compared to body dimensions, it is traditional to linearize the problem about that for a steady flow. Because of its importance to motions and seakeeping of ships (and to offshore structures operating in currents), this problem has been the subject of a large number of investigations.

The problem is classically solved by approximating the body by a distribution of singularities typically taking advantage of the slenderness (or thinness) of the body (e.g. Hanaoka 1957; Havelock 1958; Newman 1959; Maruo 1967; Ogilvie & Tuck 1969; and Newman 1978, which also contains an extensive review). Satisfactory predictions can generally be obtained except in the neighbourhood of resonance given by the frequency (ω) and forward speed (U) combination $\tau \equiv U\omega/g = \frac{1}{4}$, where g is gravitational acceleration. Despite a substantial body of work for the general linearized problem, the nature of the solution near this critical frequency for a realistic body has not been satisfactorily resolved.

For a single source, it is well known that the Green function becomes unbounded at $\tau = \frac{1}{4}$ (Haskind 1954; Wehausen & Laitone 1960). Physically, this may be explained in terms of the group velocities (in still water) of certain components of the accompanying wave which approach U as $\tau \rightarrow \frac{1}{4}$ (from below). The associated energy can no longer be radiated away, and the amplitudes of these wave components tend to grow indefinitely. Since the problem for a general body can, in principle, be represented by an appropriate distribution of such sources, it is widely accepted that the resulting seakeeping problem must likewise be singular at $\tau = \frac{1}{4}$ (e.g. Dagan & Miloh 1982). This appears also to be confirmed by existing approximate theories and calculations (e.g. Newman 1959; Wu & Eatock Taylor 1988) suggesting that this difficulty may be inherent in the linearized problem.

The present work is motivated in a large part by careful numerical calculations for the case of submerged circular and elliptical cylinders by Grue & Palm (1985) and Mo & Palm (1987). For the submerged circular cylinder, Grue & Palm (1985) offered strong numerical evidence that the amplitudes of the resonant upstream and downstream waves approach the same finite limit as $\tau \rightarrow \frac{1}{4}$. They were able to support this by examining the coefficients of an infinite set of equations which resulted from Fourier discretizations of the source strengths on the circle. Since their equations are singular at $\tau = \frac{1}{4}$, they considered the problem undetermined at this limiting value. Similar finite results were obtained for the submerged ellipse near $\tau = \frac{1}{4}$ by Mo & Palm (1987). From these results, they again reasoned (based on an integral equation similar to (3.3)), that the amplitudes should be finite as $\tau \rightarrow \frac{1}{4}$

In this paper, we offer a formal proof that a finite solution exists at $\tau = \frac{1}{4}$ for a general class of bodies. In particular, a simple necessary and sufficient geometric condition is found for such finite solutions. This condition depends on and must be satisfied for all possible values of the frequency ω but is not a function of U. When the body is submerged, the condition is satisfied if and only if the body has non-zero volume (e.g. a submerged cylinder but not a point source or dipole). For a body intersecting the free surface, sufficient conditions can be obtained by considering deviations of the body from a vertically uniform geometry of the same waterplane and draught. The resulting condition has a similar geometric interpretation to that of John (1950) in another context (the uniqueness of the solution of the floating body motion problem without forward speed).

In this paper, we concentrate only on the neighbourhood of $\delta^2 \equiv |1-4\tau| \ll 1$. The linearized boundary-value problem and the behaviour of the Green function near $\tau = \frac{1}{4}$ are reviewed in §2. We reformulate this problem as source-distribution boundary-integral equations on the body for both submerged (§3) and surface-intersecting bodies (§4) and discuss the solutions as $\tau \rightarrow \frac{1}{4}$. It is shown that the solutions are bounded for a general class of geometries satisfying an integral condition with simple geometric interpretations. As an illustration, we consider in §5 the special case of a submerged circular cylinder and obtain a closed-form (finite) solution for motions in the neighbourhood of $\tau = \frac{1}{4}$.

For simplicity and to obtain closed-form answers, we present the problem mainly in two dimensions although similar results and geometric conditions follow directly for three-dimensional bodies. This is outlined in §6. Finally, in the discussion, §7, we identify the difficulties inherent in existing approximate theories and in direct numerical solutions of the integral equations as $\tau \rightarrow \frac{1}{4}$. In the latter case, a simple remedy is provided based on an alternative form of the integral equation valid for small δ^2.

2. The boundary-value problem and Green function

We consider the generalized Kelvin–Neumann problem (Haskind 1946) of a two-dimensional body translating with constant forward speed U parallel to the undisturbed free surface in deep water while at the same time undergoing small oscillatory motion and/or encountering small-amplitude waves at frequency ω. A Cartesian coordinate system $o-xz$ is chosen fixed to the mean position of the body, with $o-x$ in the undisturbed free surface, x pointing in the direction of forward speed, and z positive upwards. The fluid is assumed inviscid and incompressible, and the motion irrotational. The flow can be described by a velocity potential:

$$\Phi^{*}(x,z,t) = \bar{\phi}(x,z) + \Phi(x,z,t) = \bar{\phi}(x,z) + \mathrm{Re}\{\phi(x,z)\mathrm{e}^{\mathrm{i}\omega t}\}, \tag{2.1}$$

where $\bar{\phi}$ is the potential due to the steady forward motion of the body, and Φ the unsteady potential associated with the body oscillations and/or incident waves. We focus on the unsteady potential Φ and do not further concern ourselves with $\bar{\phi}$ which is related to the steady wave resistance problem.

The time-independent potential ϕ satisfies Laplace's equation within the fluid and vanishes at large depth, $\nabla\phi \rightarrow 0$ as $z \rightarrow -\infty$. For small-amplitude incident waves or body motions, the linearized free-surface condition is

$$(\mathrm{i}\omega - U\frac{\partial}{\partial x})^2\phi + g\frac{\partial \phi}{\partial z} = 0 \qquad \text{on} \quad z = 0\,. \tag{2.2}$$

The kinematic boundary condition applied at the mean position of the wetted body surface, S_B, can be written as

$$\frac{\partial \phi}{\partial n} = f(x,z) \qquad \text{on} \quad S_B\,, \tag{2.3}$$

where $\boldsymbol{n} = (n_x, n_z)$ is the unit normal out of the body. In (2.3), the forcing term $f(x,z)$ is given in terms of the imposed body oscillations and incident wave as well as the so-called '$\boldsymbol{m}$-terms' due to the steady potential $\bar{\phi}$ (e.g. Newman 1978). The boundary-value problem for ϕ is complete with the imposition of an appropriate radiation condition, in this case a physical requirement that only waves with group velocity greater than (less than) the forward speed can be present far up (down) stream of the body.

At this point, we should remark that a general uniqueness theory for the boundary-value problem with the free-surface condition (2.2) is as yet unavailable. Despite this, the solution of the present problem has been pursued in a large number of studies (see, e.g. Newman 1978). For submerged bodies in steady motion, the Kelvin–Neumann problem is shown (with some restrictions, see Kochin 1937; Dern 1980) to possess a unique solution. We are unable to extend this result and simply postulate the uniqueness of the stated problem at least for the general case when $\tau \neq \frac{1}{4}$.

We define a Green function, $G(x,z;x',z')$, which is harmonic everywhere in the fluid except at (x',z') where it is source like. In addition, G satisfies the linearized free-surface condition (2.2), the radiation condition, and vanishes at large depth. Physically, G represents the potential due to a translating point source, velocity U, with a pulsating strength, frequency ω.

The solution for G was obtained by Haskind (1954), which we rewrite as follows:

$$G(x,z;x',z') = G_0 + G_1 + G_2 + G_3 + G_4, \tag{2.4}$$

where

$$G_0 = \tfrac{1}{2}\{\ln[(x-x')^2 + (z-z')^2] - \ln[(x-x')^2 + (z+z')^2]\}\,, \tag{2.5}$$

$$G_1 = \frac{\mathrm{i}\pi}{(1-4\tau)^{\frac{1}{2}}} \mathrm{e}^{k_1[-\mathrm{i}(x-x')+(z+z')]} + \frac{1}{(1-4\tau)^{\frac{1}{2}}} \int_0^\infty \frac{1}{m-k_1} \mathrm{e}^{m[-\mathrm{i}(x-x')+(z+z')]} \mathrm{d}m\,, \tag{2.6}$$

$$G_2 = \frac{\mathrm{i}\pi}{(1-4\tau)^{\frac{1}{2}}} \mathrm{e}^{k_2[-\mathrm{i}(x-x')+(z+z')]} - \frac{1}{(1-4\tau)^{\frac{1}{2}}} \int_0^\infty \frac{1}{m-k_2} \mathrm{e}^{m[-\mathrm{i}(x-x')+(z+z')]} \mathrm{d}m\,, \tag{2.7}$$

$$G_3 = \frac{-\mathrm{i}\pi}{(1+4\tau)^{\frac{1}{2}}} \mathrm{e}^{k_3[\mathrm{i}(x-x')+(z+z')]} + \frac{1}{(1+4\tau)^{\frac{1}{2}}} \int_0^\infty \frac{1}{m-k_3} \mathrm{e}^{m[\mathrm{i}(x-x')+(z+z')]} \mathrm{d}m\,, \tag{2.8}$$

$$G_4 = \frac{\mathrm{i}\pi}{(1+4\tau)^{\frac{1}{2}}} \mathrm{e}^{k_4[\mathrm{i}(x-x')+(z+z')]} - \frac{1}{(1+4\tau)^{\frac{1}{2}}} \int_0^\infty \frac{1}{m-k_4} \mathrm{e}^{m[\mathrm{i}(x-x')+(z+z')]} \mathrm{d}m\,, \tag{2.9}$$

and Cauchy principal-value integrals are indicated. In the above, we define $\tau \equiv U\omega/g$ and the four wavenumbers are defined by

$$k_{1,2} = \frac{\kappa}{8\tau^2}(1 - 2\tau \pm (1-4\tau)^{\frac{1}{2}}) \;; \qquad k_{3,4} = \frac{\kappa}{8\tau^2}(1 + 2\tau \pm (1+4\tau)^{\frac{1}{2}}) \;; \tag{2.10}$$

where $\kappa \equiv 4\omega^2/g$.

The far-field wave behaviour can be readily seen from (2.10). For $\tau < \frac{1}{4}$, all four wavenumbers are real and the k_1, k_3, and k_4 waves propagate downstream (behind the body), while the k_2 wave appears upstream. For $\tau > \frac{1}{4}$, k_3 and k_4 are still real, whereas k_1 and k_2 become complex. As a result, the k_3 and k_4 waves remain downstream, while the k_1 and k_2 waves are evanescent.

Our interest is in the neighbourhood of $\tau = \frac{1}{4}$, where k_1 and k_2 approach a common value, and G_1 and G_2 become singular. Physically, this corresponds to the k_1 and k_2 waves merging into a single wave with group velocity equal to U. For a *single* source, the energy of this wave cannot radiate away to infinity resulting in an unbounded buildup of energy, at least in the context of linearized theory (see Dagan & Miloh 1982). The key finding of this paper is that for an *actual* physical body, the wave sources of non-trivial strength may combine in such a way that the total solution remains finite as $\tau \to \frac{1}{4}$. We prove that this is indeed the case subject to a necessary and sufficient condition on the geometry of the body.

For convenience, we define $\delta^2 \equiv |1-4\tau|$. For $\delta^2 \ll 1$, we have from (2.10):

$$k_{1,2} = \kappa[1 + O(\delta)]\,, \qquad \delta^2 \ll 1 \tag{2.11}$$

In the following, we consider asymptotic expansions valid for $\kappa|\boldsymbol{x}-\boldsymbol{x}'|\delta = o(1)$ as $\delta \to 0$. Note the limit of $|\boldsymbol{x}| \to \infty$ such that $\kappa|\boldsymbol{x}-\boldsymbol{x}'|\delta \to \infty$ while $\delta \ll 1$ requires special care and is taken up in the Appendix.

From (2.6), (2.7), we write

$$G_1 + G_2 = \frac{2\pi\mathrm{i}}{\delta} \mathrm{e}^{\kappa[-\mathrm{i}(x-x')+(z+z')]} + G' + O(\delta)\,, \qquad \delta^2 \ll 1\,. \tag{2.12}$$

In (2.12), $G' = O(1)$ results from the principal-value integrals in (2.6) and (2.7), and is given by

$$\frac{1}{4}G' + 1 = \kappa[-\mathrm{i}(x-x') + (z+z')]\mathrm{e}^{\kappa[-\mathrm{i}(x-x')+(z+z')]} \int_{-\kappa}^{\infty} \frac{1}{m} \mathrm{e}^{m[-\mathrm{i}(x-x')+(z+z')]} \mathrm{d}m\,. \tag{2.13}$$

3. Submerged bodies

We construct a solution of the problem in terms of a source distribution (Brard 1972):

$$\phi(x,z) = \int_{S_B} \sigma(x',z')G(x,z;x',z')\mathrm{d}s' , \tag{3.1}$$

where $\sigma(x',z')$ is the source strength distribution on the body. Clearly, (3.1) satisfies all the conditions of the boundary-value problem except for that on the body. Imposing the body boundary condition (2.3), we obtain an integral equation for the unknown source strength σ:

$$\pi\sigma(x,z) + ⨍_{S_B} \sigma(x',z')\frac{\partial}{\partial n}G(x,z;x',z')\mathrm{d}s' = f(x,z) , \quad (x,z) \in S_B. \tag{3.2}$$

As with the original boundary-value problem, we assume that a unique solution to (3.2) exists in general for $\tau \neq \frac{1}{4}$. As $\tau \to \frac{1}{4}$, the kernel of (3.2) becomes unbounded everywhere due to the presence of G_1 and G_2. Our interest is in this neighbourhood, so that for $\delta^2 \ll 1$, we substitute the asymptotic behaviour of $G_1 + G_2$ in (2.12) into (3.2) and rewrite the integral equation as

$$\begin{aligned}\pi\sigma(x,z) &+ \frac{2\pi\kappa}{\delta}(n_x + \mathrm{i}n_z)\mathrm{e}^{\kappa(-\mathrm{i}x+z)} \int_{S_B} \sigma(x',z')\mathrm{e}^{\kappa(\mathrm{i}x'+z')}\mathrm{d}s' \\ &+ ⨍_{S_B} \sigma(x',z')\tilde{G}_n(x,z;x',z')\mathrm{d}s' = f(x,z) + O(\delta) , \qquad \delta^2 \ll 1 , \end{aligned} \tag{3.3}$$

where the principal-value integral involving $\tilde{G} \equiv G' + G_0 + G_3 + G_4$ is continuous as $\tau \to \frac{1}{4}$.

We now define the Kochin function

$$\alpha \equiv \int_{S_B} \sigma(x,z)\mathrm{e}^{\kappa(\mathrm{i}x+z)}\mathrm{d}s , \tag{3.4}$$

and rewrite (3.3) as

$$\begin{aligned}\sigma(x,z) = &- \frac{2\kappa\alpha}{\delta}(n_x + \mathrm{i}n_z)\mathrm{e}^{\kappa(-\mathrm{i}x+z)} \\ &- \frac{1}{\pi}⨍_{S_B} \sigma(x',z')\tilde{G}_n(x,z;x',z')\mathrm{d}s' + \frac{f(x,z)}{\pi} + O(\delta) , \qquad \delta^2 \ll 1 . \end{aligned} \tag{3.5}$$

The forcing function f, which is due to the incident and steady Kelvin waves as well as imposed body motions, is, in general, finite and assumed to be $O(1)$.

To determine the magnitude of α, we substitute σ in (3.5) into (3.4), and solve for α. After using the divergence theorem, we obtain

$$\alpha = \frac{\delta}{\pi(\delta + 2\mathrm{i}\kappa\Gamma)}\left[\mathscr{F} - \int_{S_B} \sigma(x',z')P(x',z')\mathrm{d}s'\right] + O(\delta^2) , \tag{3.6}$$

where the kernel P is given by

$$P(x',z') = ⨍_{S_B} \mathrm{e}^{\kappa(\mathrm{i}x+z)}\frac{\partial}{\partial n}(G' + G_0)\mathrm{d}s , \tag{3.7}$$

and the constants $\mathscr{F}$ and Γ are given by

$$\mathscr{F} = \int_{S_B} f(x,z)\mathrm{e}^{\kappa(\mathrm{i}x+z)}\mathrm{d}s, \tag{3.8}$$

$$\Gamma = \int_{S_B} (-\mathrm{i}n_x + n_z)\mathrm{e}^{2\kappa z}\mathrm{d}s. \tag{3.9}$$

$\mathscr{F}$ and P are independent of δ and can be at most $O(1)$. In (3.9), Γ is a function of the body geometry only for a given frequency κ.

Depending on the body geometry, there are now two possibilities. If $\Gamma \neq 0$, we substitute (3.6) back into (3.5) to obtain a new integral equation for σ:

$$\pi\sigma(x,z) - \frac{(n_x + \mathrm{i}n_z)}{\delta/2\kappa + \mathrm{i}\Gamma}\mathrm{e}^{\kappa(-\mathrm{i}x+z)} \int_{S_B} \sigma(x',z')P(x',z')\mathrm{d}s'$$
$$+ \int\!\!\!\!\!\!- _{S_B} \sigma(x',z')\tilde{G}_n(x,z;x',z')\mathrm{d}s' = F(x,z) + O(\delta)\,, \tag{3.10}$$

where

$$F(x,z) = f(x,z) - \mathscr{F}\frac{(n_x + \mathrm{i}n_z)}{\delta/2\kappa + \mathrm{i}\Gamma}\mathrm{e}^{\kappa(-\mathrm{i}x+z)} = O(1)\,. \tag{3.11}$$

The kernels in (3.10) are bounded and continuous as $\tau \to \frac{1}{4}$. Thus (3.10) is regular and, for sufficiently smooth S_B, has a bounded solution $\sigma = O(1)$ except possibly at an enumerable number of discrete values of κ for which the Fredholm determinant vanishes (e.g. Ursell 1968; for steady motions and two- and three-dimensional submerged bodies these are shown to be absent for sufficiently small Froude numbers, Kochin 1937). This is a difficulty associated with the general problem and not specifically with the limit $\delta \to 0$. Since our interest is in the latter, we do not consider this possibility any further. From (3.6), it is also clear that $\alpha = O(\delta)$ for $\Gamma \neq 0$.

We remark that for arbitrary geometries, (3.10) can be solved in general by direct numerical means for the finite solution. The velocity potential is finite as $\tau \to \frac{1}{4}$ and is given by

$$\phi(x,z) = \frac{2\pi\mathrm{i}\alpha}{\delta}\mathrm{e}^{\kappa(-\mathrm{i}x+z)} + \int_{S_B} \sigma(x',z')\tilde{G}(x,z;x',z')\mathrm{d}s' + O(\delta)\,, \tag{3.12}$$

which is bounded for $\Gamma \neq 0$. Note that in view of the approximation in (2.12), (3.12) is strictly valid for $\kappa|\boldsymbol{x}-\boldsymbol{x}'|\delta = o(1)$. The potential in this case is in fact finite everywhere even for $|x| \to \infty$ (see Appendix).

If $\Gamma = 0$, then from (3.6), α is at least $O(1)$. It follows from (3.5) that $\sigma = O(\delta^{-1})$ which becomes unbounded as $\delta \to 0$.

In summary, then, a finite solution to the problem exists as $\tau \to \frac{1}{4}$ if and only if

$$\Gamma \equiv \int_{S_B} (-\mathrm{i}n_x + n_z)\mathrm{e}^{2\kappa z}\mathrm{d}s \neq 0\,, \tag{3.13}$$

which is a condition that depends on the geometry S_B and the frequency $\kappa = 4\omega^2/g$ only. If $\Gamma = 0$ for any frequency κ, then a forward speed $U^2 = g/4\kappa$ can always be found for which the solution becomes unbounded. Physically, (3.13) represents a requirement that the resonant wave components must not be orthogonal to the body boundary condition.

With the use of the divergence theorem, we obtain immediately

$$\Gamma = 2\kappa \iint_B \mathrm{e}^{2\kappa z}\mathrm{d}S\,, \tag{3.14}$$

where B is the (mean) body section. Since the integrand in (3.14) is positive definite, $\Gamma \neq 0$ if and only if the (submerged) body has non-zero cross-section area. The known singular solution for a point source turns out to be a special case of $\Gamma = 0$.

4. Surface-intersecting bodies

An analogous result can be obtained for the case where the body intersects the free surface. We assume (locally) vertical intersections, and again write the potential in terms of a body surface source distribution (e.g. Ursell 1980)

$$\phi(x,z) = \int_{S_B} \sigma(x',z')G(x,z;x',z')\mathrm{d}s' - a[\sigma_- G(x,z;x_-,0) + \sigma_+ G(x,z;x_+,0)] , \quad (4.1)$$

where $a \equiv U^2/g$, and $\sigma_\pm$ represent the source strengths at the two intersection points, $x = x_\pm$.

For $\delta^2 \ll 1$, we proceed as before and write

$$\pi\sigma(x,z) = -\frac{2\pi\kappa\alpha'}{\delta}(n_x + \mathrm{i}n_z)\mathrm{e}^{\kappa(-\mathrm{i}x+z)} - \fint_{S_B} \sigma(x',z')\tilde{G}_n(x,z;x',z')\mathrm{d}s'$$
$$+ a[\sigma_- \tilde{G}_n(x,z;x_-,0) + \sigma_+ \tilde{G}_n(x,z;x_+,0)] + f(x,z) + O(\delta) , \quad (4.2)$$

where the Kochin function α' is defined to be

$$\alpha' = \int_{S_B} \sigma(x,z)\mathrm{e}^{\kappa(\mathrm{i}x+z)}\mathrm{d}s - a[\sigma_- \mathrm{e}^{\mathrm{i}\kappa x_-} + \sigma_+ \mathrm{e}^{\mathrm{i}\kappa x_+}] . \quad (4.3)$$

Again, it is clear from (4.2) that $\sigma = O(1)$ if $\alpha' \leqslant O(\delta)$. Otherwise, σ becomes unbounded as $\tau \to \frac{1}{4}$.

Substituting (4.2) into (4.3), we have

$$\alpha' = \frac{\delta}{(\delta + 2\mathrm{i}\kappa\Gamma)\pi}[\mathscr{F} + \mathscr{H} - \int_{S_B} \sigma(x',z')Q(x',z')\mathrm{d}s'] + O(\delta^2) , \quad (4.4)$$

where the kernel Q is given by

$$Q(x',z') = \fint_{S_B} \tilde{G}_n(x,z;x',z')\mathrm{e}^{\kappa(\mathrm{i}x+z)}\mathrm{d}s , \quad (4.5)$$

and the constant $\mathscr{H}$ is defined as

$$\mathscr{H} = \int_{S_B} a[\sigma_- \tilde{G}_n(x,z;x_-,0) + \sigma_+ \tilde{G}_n(x,z;x_+,0)]\mathrm{e}^{\kappa(\mathrm{i}x+z)}\mathrm{d}s - \pi a[\sigma_- \mathrm{e}^{\mathrm{i}\kappa x_-} + \sigma_+ \mathrm{e}^{\mathrm{i}\kappa x_+}]. \quad (4.6)$$

$\mathscr{F}$, $\mathscr{H}$ and Q are independent of δ and can be at most $O(1)$. If $\Gamma = 0$, $\alpha' = O(1)$ and $\sigma = O(\delta^{-1})$, and no finite solution exists as $\tau \to \frac{1}{4}$. If $\Gamma \neq 0$, $\alpha' = O(\delta)$ and $\sigma = O(1)$ and we may substitute (4.4) back into (4.2) to obtain a new integral equation for σ:

$$\pi\sigma(x,z) - \frac{(n_x + \mathrm{i}n_z)}{\delta/2\kappa + \mathrm{i}\Gamma}\mathrm{e}^{\kappa(-\mathrm{i}x+z)} \int_{S_B} \sigma(x',z')Q(x',z')\mathrm{d}s'$$
$$+ \fint_{S_B} \sigma(x',z')\tilde{G}_n(x,z;x',z')\mathrm{d}s'$$
$$- a[\sigma_- \tilde{G}_n(x,z;x_-,0) + \sigma_+ \tilde{G}_n(x,z;x_+,0)] = F(x,z) + H(x,z) + O(\delta) , \quad (4.7)$$

where

$$H(x,z) = -\frac{n_x + \mathrm{i}n_z}{\delta/2\kappa + \mathrm{i}\Gamma}\mathrm{e}^{\kappa(-\mathrm{i}x+z)}\mathscr{H} \quad (4.8)$$

Now, every term in (4.7) is finite as $\tau \to \frac{1}{4}$, so that (4.7) is regular and a bounded solution for σ can be obtained, after which the Kochin function α' can be determined

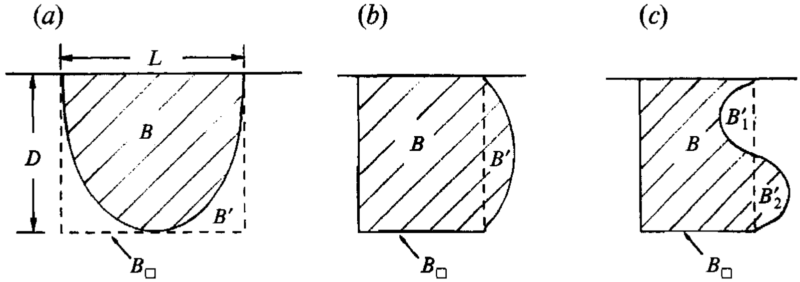

FIGURE 1. Geometric condition for a body intersecting the free surface. (a) $B' < 0$; (b) $B' > 0$; (c)$-B_2' = B_1' < 0$.

from (4.4). The problem is thus solved with

$$\phi(x,z) = \frac{2\pi i \alpha'}{\delta} e^{\kappa(-ix+z)} + \int_{S_B} \sigma(x',z')\tilde{G}(x,z;x',z')ds' \\ - a[\sigma_- \tilde{G}(x,z;x_-,0) + \sigma_+ \tilde{G}(x,z;x_+,0)] + O(\delta)\,, \tag{4.9}$$

which is finite.

As with the submerged case, the necessary and sufficient condition for a finite solution is (3.13), i.e. $\Gamma \neq 0$. Use of the divergence theorem here yields

$$\Gamma = 2\kappa \iint_B e^{2\kappa z} dS - L\,, \tag{4.10}$$

where $L = x_+ - x_- > 0$ is the waterline width of the body. Let us divide the mean body section B into two parts: $B = B_\square + B'$, where $B_\square$ is the rectangle with width L and depth D equal to the maximum draught of the body, and B' the difference between B and B (see figure 1). The double integral over $B_\square$ can be evaluated yielding

$$\Gamma = 2\kappa \iint_{B'} e^{2\kappa z} dS - Le^{-2\kappa H}. \tag{4.11}$$

If the body B is completely enclosed by $B_\square$, B' is negative and so also is integral over B' in (4.11) negative. Whence Γ is negative definite and $B \subset B_\square$ is a sufficient condition for (3.13).

If $B \not\subset B_\square$ (for example, figure 1*b*), the integral over B' may be positive, and a value of κ may exist for which $\Gamma = 0$. To illustrate this further, consider the case of a circular cylinder, radius R, which intersects the free surface (for simplicity still assuming the body to be *locally* vertical at the intersection points). If the centre of the cylinder z_c is above the free surface, $z_c > 0$, then $B \subset B_\square$ and Γ is negative definite. If the cylinder is completely submerged, $z_c < -R$, then from (3.14), Γ is positive definite. For the intermediate case of $-1 < z_c/R < 0$, however, (4.11) shows that Γ is negative for $\kappa = 0$ but increases monotonically with κ and eventually changes sign. For any $z_c/R \in (-1,0)$, there exists a particular value of the frequency $\kappa = \kappa_0 > 0$ for which $\Gamma = 0$. It follows that a finite solution does not exist at that frequency and at a forward speed corresponding to $\tau = \frac{1}{4}$ given by $U_0^2 = g/4\kappa_0$. Figure 2 shows a plot of $\kappa_0 R$ as a function of $\beta \equiv \sin^{-1}(-z_c/R)$ for this case. Note that $\kappa_0 R \sim -\ln\beta/\beta$ as $\beta \to 0$.

The sufficient condition on the geometry, $B \subset B_\square$, is similar to that of John (1950) for the motion of a floating body (without forward speed) which requires that for

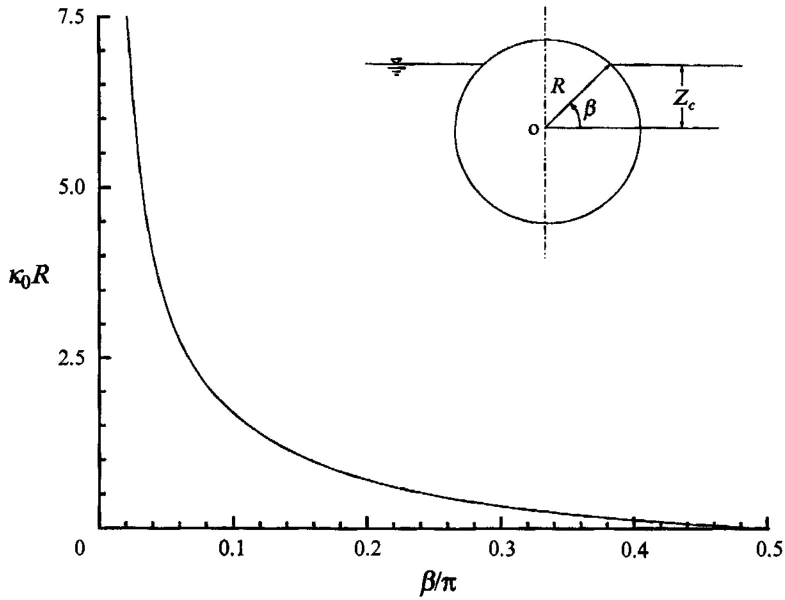

FIGURE 2. Dimensionless frequency $\kappa_0 R$ for $\Gamma = 0$ as a function of the submergence of a floating circular cylind r.

every point of the mean free surface (in this case $x \notin [x_-, x_+]$), the entire vertical segment below it must not intersect the mean body. The actual requirement of $\Gamma \neq 0$ is, however, more general (less restrictive) and admits, for example, a geometry such as that depicted in figure 1(*c*).

5. Application to a submerged circular cylinder

As an illustration, we consider the special case of a translating and oscillating submerged circular cylinder near $\tau = \frac{1}{4}$. Grue & Palm (1985) investigated this problem computationally using a source distribution (cf. (3.2)) represented by Fourier series. They obtained solutions very close to $\tau = \frac{1}{4}$ although the kernel of their integral equation becomes everywhere singular as $\tau \to \frac{1}{4}$ (cf. (3.3)). In this section, we obtain the finite solution to this problem in the neighbourhood of $\tau = \frac{1}{4}$ ($\delta^2 \ll 1$). In particular, we provide closed-form asymptotic solutions for the far-field amplitudes of the k_1 and k_2 waves (which have a common finite value at $\tau = \frac{1}{4}$).

A local cylindrical coordinate system (r, θ) is placed at the centre of the cylinder, which is at a depth h below the mean water level. Thus, $r^2 = x^2 + (z+h)^2$ and θ is measured counterclockwise from positive x. The geometry parameter Γ for a circular cylinder can be found in closed form

$$\Gamma = 2\pi R e^{-2\kappa h} I_1(2\kappa R) , \tag{5.1}$$

in which R is the radius of the cylinder and I_1 the modified Bessel function of the first kind.

For a circular cylinder, we can easily prove the following relationship:

$$⨍_{S_B} \sigma(x', z') \frac{\partial G_0}{\partial n} \mathrm{d}s' = \int_{S_B} \sigma(x', z') \mathrm{d}s' . \tag{5.2}$$

Since no fluid crosses the surface of the rigid body, the net source vanishes. As a result, there is no G_0 term in $\tilde{G}$ nor in the kernel P. Given the forcing function F, the solution to the integral equation (3.10) must, in general, be obtained numerically. For relatively deep submergence, κh, however, the problem simplifies. In particular, the amplitudes of the $k_{1,2}$ waves can be obtained in closed form and interestingly do not explicitly depend on the source strength σ.

The Kochin function, α, is calculated from (3.6). Since the kernel P (without G_0) diminishes with submergence κh like $e^{-3\kappa h}$ (cf. (3.7)), the second term in (3.6) can be neglected for large submergence:

$$\alpha \approx \frac{\mathscr{F}\delta}{\pi(\delta + 2i\kappa\Gamma)} \tag{5.3}$$

which is $O(e^{\kappa h})$ since $\Gamma = O(e^{-2\kappa h})$ from (5.1). To determine the potential, we substitute $\tilde{G} = G' + G_3 + G_4$ into (3.12). Since G' diminishes as $e^{-\kappa h}$ for sources on the cylinder, its contribution to the κ wave is small compared to that due to α which is proportional to $e^{\kappa h}$. The potential field is then given by

$$\phi(x,z) = \frac{2i\mathscr{F}}{\delta + 2i\kappa\Gamma} e^{\kappa(-ix+z)} + \int_{S_B} \sigma(x',z')(G_3 + G_4)ds' + O(\delta) . \tag{5.4}$$

From the dynamic free-surface condition, the surface elevation η is given by

$$\eta(x) = -\frac{1}{g}(i\omega - U\frac{\partial}{\partial x})\phi(x,0) . \tag{5.5}$$

The wave elevations far upstream and downstream of the body are

$$\eta = A_2 e^{-ik_2 x} , \qquad x \to +\infty , \tag{5.6}$$

$$\eta = A_1 e^{-ik_1 x} + A_3 e^{ik_3 x} + A_4 e^{ik_4 x} , \qquad x \to -\infty , \tag{5.7}$$

with the wave amplitudes given from (5.4) by

$$A_{1,2} = 2\mathscr{F} \frac{(\omega + Uk_{1,2})}{g(\delta + i2\kappa\Gamma)} + O(\delta) , \tag{5.8}$$

$$A_{3,4} = \mp \frac{2\pi(\omega - Uk_{3,4})}{g(1+4\tau)^{\frac{1}{2}}} \int_{S_B} \sigma(x,z) e^{k_{3,4}(-ix+z)} ds + O(\delta) . \tag{5.9}$$

From (5.8), it is clear that $A_{1,2}$ are independent of the source strength σ. Thus, the amplitudes of the k_1 and k_2 waves are explicit and do not require the solution of the integral equation (3.10).

In principle, it is necessary to solve the steady problem first to provide for the body boundary condition $f(x,z)$. Again for relatively deep submergence, we neglect the free-surface effect and write the potential for steady flow past the circular cylinder as that around a dipole

$$\bar{\phi}(x,z) = -Ux\left(1 + \frac{R^2}{r^2}\right) \tag{5.10}$$

Considering only the radiation problem, f is then given by

$$f(\theta) = \zeta_x \left(i\omega\cos\theta + \frac{2U}{R}\cos 2\theta \right) + \zeta_z \left(i\omega \sin\theta + \frac{2U}{R}\sin 2\theta \right) , \tag{5.11}$$

where ζ_x and ζ_z are respectively the amplitudes of the sway and heave motions of the body.

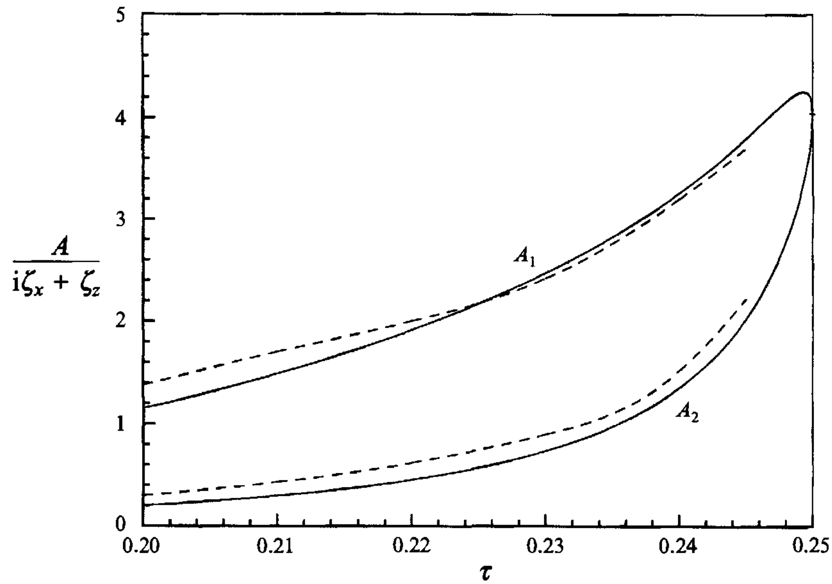

FIGURE 3. Amplitudes of the k_1 (upper branch) and k_2 (lower branch) waves radiated by the heave and sway oscillations of a submerged circular cylinder as a function of $\tau \equiv U\omega/g$. Asymptotic solution (5.13) (——); direct numerical calculations (Grue & Palm 1985) (- - -). ($F_r = U/(gR)^{\frac{1}{2}}$=0.4, h/R=2).

For the calculation of the coefficient $\mathscr{F}$, we first replace (x, z) by the corresponding cylindrical coordinates (r, θ), and expand the exponential in (3.8) in Taylor series. The integration over θ can be readily carried out yielding

$$\mathscr{F} = \pi\kappa R^2 e^{-\kappa h}(\omega + \kappa U)(-\zeta_x + i\zeta_z)\,, \tag{5.12}$$

which is the limiting value for $\mathscr{F}$ with $\delta^2 \ll 1$. For somewhat larger δ, the accuracy of (5.12) is improved by simply replacing the wavenumber κ with $k_{1,2}$ respectively for $A_{1,2}$. (This is equivalent to factoring out $e^{k_{1,2}z}$ rather than $e^{\kappa z}$ in (2.12).) Substituting $\mathscr{F}$ and Γ into (5.8), we obtain finally

$$\frac{A_{1,2}}{i\zeta_x + \zeta_z} = \frac{2\pi k_{1,2} R e^{-k_{1,2}h}}{-i(1-4\tau)^{\frac{1}{2}} + 4\pi\kappa R I_1(2\kappa R) e^{-2\kappa h}} \left(\frac{1 \pm (1-4\tau)^{\frac{1}{2}}}{2F_r}\right)^2 + O(\delta)\,, \tag{5.13}$$

where $F_r \equiv U/(gR)^{\frac{1}{2}}$ is the Froude number.

Equation (5.13) is consistent with the known result for a submerged circular cylinder that the far-field waves generated by unit sway or heave motion have the same amplitude but are shifted in phase by $\frac{1}{2}\pi$.

Figure 3 plots (5.13) for $A_{1,2}$ as a function of τ for the parameters $h/R = 2$ and $F_r = 0.4$. The limiting value of $A_{1,2}/(i\zeta_x + \zeta_z)$ as $\tau \to \frac{1}{4}$ is 4.018.... These parameter values for h/R and F_r coincide with one of the two cases computed by Grue & Palm (1985) for which they provide values for τ very close to $\frac{1}{4}$. For comparison, their numerical values are reproduced in figure 3. The comparison both in terms of the magnitudes and asymptotic slopes is quite satisfactory for this moderate submergence.

Finally, we consider the value of $A_{1,2}$ at $\tau = \frac{1}{4}$ as a function of wave frequency κR

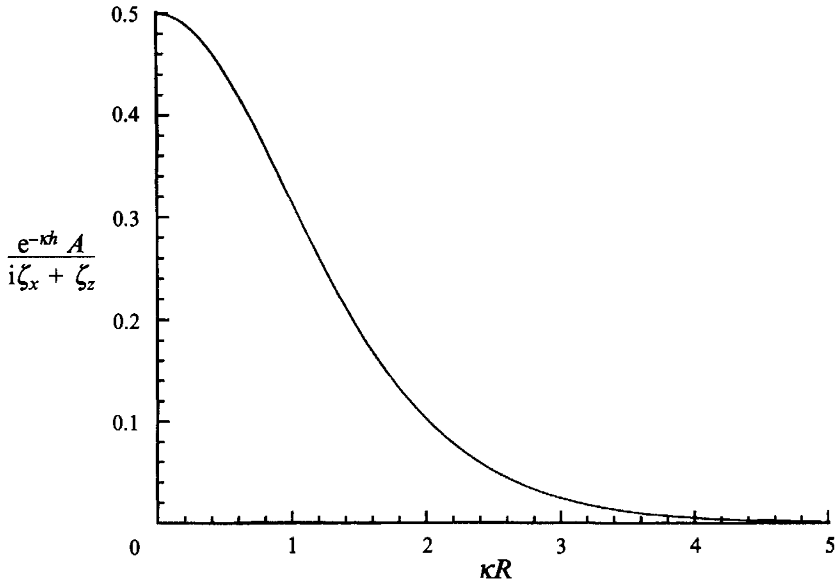

FIGURE 4. Limiting amplitude at $\tau \equiv U\omega/g = \frac{1}{4}$ of the $k_{1,2}$ waves due to the forced heave and sway oscillations of a submerged circular cylinder as a function of the dimensionless frequency κR. ($h/R = 2$).

($= (2F_r)^{-2}$). Evaluating (5.13) at $\tau = \frac{1}{4}$, we obtain

$$\frac{A_{1,2}}{(i\zeta_x + \zeta_z)} = \frac{\kappa R e^{\kappa h}}{2I_1(2\kappa R)}. \tag{5.14}$$

Figure 4 shows a plot of this limiting amplitude normalized by $e^{-\kappa h}$. As a check, at the other value of $F_r = 1.0$ ($\kappa R = 0.25$) computed by Grue & Palm (1985), the extrapolated value at $\tau = \frac{1}{4}$ from their curves again agrees well with the value of $A_{1,2}/(i\zeta_x + \zeta_z) = 0.799...$ given by (5.14). For low frequency (and large U), (5.14) has the limit of $\frac{1}{2}$ as $\kappa R \to 0$ — a surprisingly simple result. For high frequency, $\kappa R \gg 1$, the amplitudes vanish exponentially, $A_{1,2}e^{-\kappa h}/(i\zeta_x + \zeta_z) \sim \pi^{\frac{1}{2}}(\kappa R)^{\frac{3}{2}}e^{-2\kappa R}$.

6. Generalization to three dimensions

The foregoing analyses and results can be generalized to three dimensions. The key requirement is the separability of the dependence on $\boldsymbol{x}$ and $\boldsymbol{x}'$ in the leading-order term of the Green function for $\delta^2 \ll 1$ (cf. (2.12)) leading to the factoring of the Kochin functions α and α' (cf. (3.5), (4.2)).

The three-dimensional Green function for this problem (e.g. Wehausen & Laitone 1960) can be rewritten for $\tau < \frac{1}{4}$ as

$$G(x, y, z; x', y', z') = \frac{1}{r} - \frac{1}{r_1} + \frac{2}{\pi}\int_0^{\frac{1}{2}\pi} \frac{d\theta}{(1 - 4\tau\cos\theta)^{\frac{1}{2}}} \int_0^\infty \left(\frac{1}{k - k_1} - \frac{1}{k - k_2}\right) h(\theta, k)dk$$
$$+ \frac{2}{\pi}\int_{\frac{1}{2}\pi}^{\pi} \frac{d\theta}{(1 - 4\tau\cos\theta)^{\frac{1}{2}}} \int_0^\infty \left(\frac{1}{k - k_3} - \frac{1}{k - k_4}\right) h(\theta, k)dk, \tag{6.1}$$

in which

$$\begin{matrix} r \\ r_1 \end{matrix} = [(x-x')^2+(y-y')^2+(z\mp z')^2]^{\frac{1}{2}}\,, \tag{6.2}$$

$$h(\theta,k) = k\cos[k(y-y')\sin\theta]\mathrm{e}^{k[(z+z')-\mathrm{i}(x-x')\cos\theta]}\,, \tag{6.3}$$

$$k_{1,2} = \kappa\frac{1-2\tau\cos\theta\mp(1-4\tau\cos\theta)^{\frac{1}{2}}}{8\tau^2\cos^2\theta}\,. \tag{6.4}$$

The wavenumbers k_3 and k_4 have solutions of the same form as k_1 and k_2. To satisfy the radiation condition at infinity, the integration paths for k over the singularities in (6.1) are defined as $k_1-\mathrm{i}\epsilon$, $k_2+\mathrm{i}\epsilon$, $k_3-\mathrm{i}\epsilon$, $k_4-\mathrm{i}\epsilon$ as $\epsilon\to 0_+$. After the use of the Plemelj formula, the k integrals reduce to Cauchy principle-value integrals plus the contribution from the four singularities at k_1, k_2, k_3 and k_4. As $\tau\to\frac{1}{4}$, the term $(1-4\tau\cos\theta)^{-\frac{1}{2}}$ becomes unbounded along $\theta=0$. Thus G is dominated by the integration near $\theta=0$, i.e.

$$\begin{aligned} G(x,y,z;x',y',z') = {} & \frac{2}{\pi}\int_0^{\epsilon}\frac{\mathrm{d}\theta}{(1-4\tau\cos\theta)^{\frac{1}{2}}}\,⨍_0^{\infty}\left(\frac{1}{k-k_1}-\frac{1}{k-k_2}\right)h(\theta,k)\mathrm{d}k \\ & -2\mathrm{i}\int_0^{\epsilon}\frac{h(\theta,k_1)+h(\theta,k_2)}{(1-4\tau\cos\theta)^{\frac{1}{2}}}\mathrm{d}\theta+O(1) \qquad \text{as } \tau\to(\tfrac{1}{4})_-\,. \end{aligned} \tag{6.5}$$

It can be shown that the double integral in (6.5) remains finite as $\tau\to\frac{1}{4}$. By expanding $\cos\theta$ in Taylor series about $\theta=0$, the single integral can be carried out yielding finally

$$G(x,y,z;x',y',z') = \mathrm{i}8\sqrt{2}\kappa\ln(1-4\tau)^{\frac{1}{2}}\mathrm{e}^{\kappa[(z+z')-\mathrm{i}(x-x')]}+O(1) \qquad \text{as } \tau\to(\tfrac{1}{4})_-\,. \tag{6.6}$$

The result is identical for $\tau\to(\frac{1}{4})_+$ and can be obtained similarly by considering this limit for the expression of G for $\tau>\frac{1}{4}$.

We now note that the dependence of G on $\boldsymbol{x}$, $\boldsymbol{x}'$ in (6.6) is identical to (2.12) for the two-dimensional case. The analyses in §§2, 3 thus follow directly leading to geometric conditions (3.14) and (4.10) for submerged and surface-intersecting bodies respectively. The integrands remain *identical*, but now the integrals are to be performed over the mean *two-dimensional* surface of the body. For (4.10), the waterline width L is now replaced by the waterplane area of the body.

7. Discussion

The present findings can be motivated somewhat by physical arguments. Although the single source (Green function) becomes unbounded everywhere as $\tau\to\frac{1}{4}$, the distribution of such sources on the body satisfies a finite forcing. Physically, this requires that the Kochin function α/δ, which measures the net contribution of the sources at a fixed point, remains finite (i.e. $\alpha\leqslant O(\delta)$) as $\delta\to 0$. The necessary and sufficient condition for this to be true for a given body is the geomtric condition $\Gamma\neq 0$, a function of the frequency κ but not of U. We reason that $\Gamma\neq 0$ is in effect a requirement that the Green function (in fact just the resonant $k_{1,2}$ waves) is not orthogonal to the boundary condition on the body.

We note that the present problem is a classical one for which a number of approximate theories (e.g. Havelock 1958; Newman 1959; Dagan & Miloh 1981) exist, all of which indicate that the solution to the problem is singular as $\tau\to\frac{1}{4}$. The apparent contradiction with the present finding turns out to be the result of a common feature of the existing theories, namely, that the body boundary condition is enforced only in an approximate manner.

Consider, for example, a submerged circular cylinder represented by a single dipole Green function G_x at the centre. For τ not near $\frac{1}{4}$, the error in the normal velocity on the body surface is $\epsilon \sim e^{-2\kappa h}$ which vanishes as the body submergence increases. For $\delta^2 \ll 1$, however, $\epsilon \sim e^{-2\kappa h}/\delta$ since $G_{xn} \sim O(\delta^{-1})$, and the approximation is unacceptable as $\tau \to \frac{1}{4}$ for any finite κh. Interestingly, for a point-like body, which may be a valid approximation for a very deeply submerged object, $\Gamma = 0$ according to (3.14). The solution at $\tau = \frac{1}{4}$ is then in fact unbounded and is consistent with existing results.

It is noteworthy that existing numerical solutions to this problem (e.g. Grue & Palm 1985; Wu & Eatock Taylor 1988) have likewise met with difficulties close to $\tau = \frac{1}{4}$. The computational difficulty arises from a direct solution of the integral equation (3.2) as $\tau \to \frac{1}{4}$. From (3.3) and dividing by $(n_x + in_z)e^{\kappa(-ix+z)}/\delta$, we have

$$2\pi\kappa \int_{S_B} \sigma(x',z')e^{\kappa(ix'+z')}ds' + \frac{e^{\kappa(ix-z)}\delta}{n_x + in_z}\left[\pi\sigma(x,z) + \fint_{S_B} \sigma(x',z')\tilde{G}_n(x,z;x',z')ds'\right] = \frac{e^{\kappa(ix-z)}\delta}{n_x + in_z}f(x,z) + O(\delta^2)\,. \quad (7.1)$$

In a typical numerical solution, (7.2) is discretized by subdividing S_B into N segments, and local basis functions are assumed for the source strength σ over each segment, say resulting in N unknown values for σ. Equation (7.2) is then collocated at N selected points (say one in each segment) resulting in a system of N linear equations for the N unknowns. The resulting coefficient matrix may be formally expressed as

$$([A_1] + [A_2]\delta) + O(\delta^2)\,, \quad (7.2)$$

where $[A_1]$ and $[A_2]$ are the $N \times N$ influence matrices corresponding to the first and second terms respectively on the left-hand side of (7.2) and are formally independent of δ. As $\tau \to \frac{1}{4}$, (7.2) reduces to $[A_1] + O(\delta)$. From (7.2), it is clear that $[A_1]$ is not a function of the field point $\boldsymbol{x}$. Thus, the coefficient matrix has identical rows regardless of the position of the collocation points and is singular. The nature of the computational difficulty in the solution of (7.2) for $\delta^2 \ll 1$ is hence clear.

It is useful to point out that our analysis in §§2, 3 provides a simple remedy for the computational problem. For τ near the critical frequency, the numerical difficulties are easily avoided by solving the regular equations (3.10) for a submerged body or (4.7) for a surface-piercing body instead of the singular equation (3.2).

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Appendix. The velocity potential at large distances

In this appendix, we consider the behaviour of the solution in the limit $|\boldsymbol{x}| \to \infty$ such that $\kappa|\boldsymbol{x}|\delta \to \infty$ while $\delta \ll 1$. For simplicity, we consider the case of a two-dimensional submerged body.

As $|\boldsymbol{x}| \to \infty$, G_0 vanishes and the principal-value integrals in (2.6) to (2.9) can be integrated analytically via contour integration. For $x \to +\infty$, G_1, G_3, and G_4 vanish, while

$$G_2 \sim \frac{i2\pi}{(1-4\tau)^{\frac{1}{2}}}e^{k_2[-i(x-x')+(z+z')]}\,. \quad (A\,1)$$

For $x \to -\infty$, G_2 vanishes, while

$$G_1 \sim \frac{\mathrm{i}2\pi}{(1-4\tau)^{\frac{1}{2}}} \mathrm{e}^{k_1[-\mathrm{i}(x-x')+(z+z')]}, \tag{A 2}$$

$$G_{3,4} \sim \frac{\mp \mathrm{i}2\pi}{(1+4\tau)^{\frac{1}{2}}} \mathrm{e}^{k_{3,4}[\mathrm{i}(x-x')+(z+z')]}. \tag{A 3}$$

Substitution of (A1)–(A3) into (3.1) gives

$$\phi(x,z) \sim \frac{\mathrm{i}2\pi}{\delta} \mathrm{e}^{k_2(-\mathrm{i}x+z)} \int_{S_B} \sigma(x',z') \mathrm{e}^{k_2(\mathrm{i}x'+z')} \mathrm{d}s' \tag{A 4}$$

for $x \to +\infty$, and

$$\phi(x,z) \sim \frac{\mathrm{i}2\pi}{\delta} \mathrm{e}^{k_1(-\mathrm{i}x+z)} \int_{S_B} \sigma(x',z') \mathrm{e}^{k_1(\mathrm{i}x'+z')} \mathrm{d}s' + \int_{S_B} \sigma(x',z')(G_3+G_4)\mathrm{d}s' \tag{A 5}$$

for $x \to -\infty$. In the neighbourhood of $\delta \ll 1$, we expand the kernel $\mathrm{e}^{k_{1,2}(\mathrm{i}x'+z')}$ in Taylor series about $k_{1,2} = \kappa$:

$$\mathrm{e}^{k_{1,2}(\mathrm{i}x'+z')} = \mathrm{e}^{\kappa(\mathrm{i}x'+z')}[1 \pm 2\kappa(\mathrm{i}x'+z')\delta + O(\delta^2)], \qquad \delta^2 \ll 1. \tag{A 6}$$

After substituting (A 6) into (A 4) and (A 5), we obtain

$$\phi(x,z) \sim \mathrm{i}2\pi(\alpha/\delta - \gamma)\mathrm{e}^{k_2(-\mathrm{i}x+z)} + O(\delta) \tag{A 7}$$

for $x \to +\infty$, and

$$\phi(x,z) \sim \mathrm{i}2\pi(\alpha/\delta + \gamma)\mathrm{e}^{k_1(-\mathrm{i}x+z)} + \int_{S_B} \sigma(x',z')(G_3+G_4)\mathrm{d}s' + O(\delta) \tag{A 8}$$

for $x \to -\infty$. Here, the constant γ is given by

$$\gamma = 2\kappa \int_{S_B} (\mathrm{i}x'+z')\sigma(x',z')\mathrm{e}^{\kappa(\mathrm{i}x'+z')}\mathrm{d}s', \tag{A 9}$$

and can formally be at most $O(1)$ for finite σ. For $\Gamma \neq 0$, $\alpha/\delta = O(1)$ and $\sigma = O(1)$. Thus the potentials in (A 7) and (A 8) are bounded as $\delta \to 0$.

We remark that the $k_{1,2}$ potentials in (A 8) and (A 7) respectively approach the same finite limit as $\delta \to 0$. This is due to the fact that γ is $O(\delta)$ which can be shown starting from just before (2.12). The analysis itself is a detail and is omitted here.

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